A Study of Triple Sumudu Transform for Solving Partial Differential Equations with Some Applications

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Abstract. In this paper, we have studied the extension of the single Sumudu transform to deal with the functions in three variables. Furthermore, we have studied the main properties of triple Sumudu transform. In addition, we have applied triple Laplace transform with its main characteristics for solving several examples. Finally, we have examined the solutions for general second-order PDEs which containing three variables by using triple Laplace and Sumudu transforms.

Key words: Sumudu Transform; Laplace Transform; Triple; Partial Differential Equations.

Introduction

Partial differential equation (PDEs) plays a very important role in mathematics and the other fields of sciences because these linear or nonlinear PDEs describe the physical phenomena. Thus, it is important to know how to solve these PDEs. A number of numerical or analytical methods of solutions can be used to find the solutions of DEs. The numerical methods have provided the approximated solution of differential equations (DE) rather than the analytical solutions of the problems of the study. In most times it may be difficult to solve these DEs analytically and thus are commonly solved by integral transforms such as Laplace and Fourier transforms and the advantage of these two methods lies in their ability to transform DEs into algebraic equations, which allow a simple way to find the solutions. In Eltayeb and Kiliçman (2013), Eltayeb et al. (2012: 47); Kiliçman and Eltayeb (2008: 1124), Kiliçman and Gadain (2010) extended the concept of Laplace transform to double Laplace transform and this new operator has been widely used to solve some kinds of DEs. Then, the concept of triple Laplace transform was used to solve third-order PDEs. In addition, the properties and the applications to DEs have been determined and studied (Atangana, 2013; Khan et al., 2019; Shiromani, 2013: 848). As we see, the integral transform method is an effective way to solve some certain DEs. Thus, in the literature there are a lot of works on the theory and applications of Laplace, Fourier, Mellin and other integral transforms (Debnath and Bhatta, 2006). A little on the power series transformation such as Sumudu transform, maybe because it is little known and not widely used yet. Sumudu transform was proposed by Watugala (1993: 35) for solving DEs and control engineering problems. Among the other integral transforms, Sumudu transform has units preserving properties and thus may be used to solve problems without resorting them to the frequency domain and this is one of many strength points of this new transform. However, Belgacem et al. (2003: 103) extended the theory and the applications of Sumudu transform and applied it for fractional integrals, derivatives, and used it to solve initial value fractional differential equations (FrDEs) and (Asiru, 2002: 441) further developed this new transform and most of its fundamental properties and applied it for special functions and used it to solve fractional integrals, derivatives, and initial value FrDEs. Since then, many researchers have studied Sumudu transform and its properties. Series of papers have been published started with Belgacem and Karaballi (2006), where he extended the theory and the applications of the Sumudu transform and use it to solve the FrDEs by direct integration methods, study and prove most of Sumudu transform properties study the Laplace-Sumudu transforms duality and the complex inversion formula and avails the readers with the most comprehensive list of function transforms in the literature, up to date Kiliçman and Gadain (2010: 10) studied the extension of double Sumudu transform to triple Sumudu transform briefly and study some of its properties and its relation with triple Laplace transform. Haydar (2009: 33) studied the extension of single Sumudu transform to n-dimensions Sumudu transform and studied its main properties and gives a table of n-dimensions Sumudu transform for the most familiar functions. In this paper, we have studied the extension of the concept of Sumudu transform into triple Sumudu transform along with its main properties, studied triple Laplace transform with its main characteristics and explained them by several examples.

Preliminary

Here, we give the definition for triple Sumudu transform with the properties of triple Sumudu and Laplace transforms.

The Triple Sumudu Transform

In this subsection, we give the definition for triple Sumudu transform with the properties of triple Sumudu and Laplace transforms.

*Definition***.** The triple Sumudu transform can be defined by:

$$F(u,v,c) = S_3[f(x,y,t);(u,v,c)] = \frac{1}{uvc}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{x}{u}+\frac{y}{v}+\frac{t}{c})} f(x,y,t)dxdydt$$

or by,

$$F(u,v,c) = S_3[f(x,y,t);(u,v,c)] = \int_0^\infty\int_0^\infty\int_0^\infty e^{-(x+y+t)} f(ux,vy,ct)dxdydt$$

Properties of the Triple Sumudu Transform

1. The triple Sumudu transform of the third partial derivative with respect to **x** has the following form:

$$\begin{aligned} S_3[\frac{\partial^3 f(x,y,t)}{\partial x^3} \quad ; \quad (u,v,c)] &= \frac{1}{uvc}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{x}{u}+\frac{y}{v}+\frac{t}{c})}\frac{\partial^3 f(x,y,t)}{\partial x^3}dxdydt \\ &= \frac{1}{vc}\int_0^\infty e^{-(\frac{t}{v}+\frac{t}{c})}(\frac{1}{u}\int_0^\infty e^{-\frac{x}{u}}\frac{\partial^3 f(x,y,t)}{\partial x^3}dx)dydt \qquad (1)\end{aligned}$$

The integral inside the brackets can be computed individually as follows:

$$\frac{1}{u}\int_0^\infty e^{-\frac{x}{u}}\frac{\partial^3 f(x,y,t)}{\partial x^3}dx = \frac{1}{u^3}F(u,y,t) - \frac{1}{u^3}f(0,y,t) - \frac{1}{u^2}\frac{\partial f(0,y,t)}{\partial x} - \frac{1}{u}\frac{\partial^2(0,y,t)}{\partial x^2}$$

2. By taking Sumudu transform with respect to y, we get the double Sumudu transform as follows:

$$\frac{1}{v}\int_0^\infty e^{-\frac{y}{v}}\frac{\partial^3 f(x,y,t)}{\partial y^3}dx = \frac{1}{v^3}F(x,v,t) - \frac{1}{v^3}f(x,0,t) - \frac{1}{v^2}\frac{\partial f(x,0,t)}{\partial y} - \frac{1}{v}\frac{\partial^2(x,0,t)}{\partial y^2}$$

3. By taking Sumudu transform with respect to t ,we get the double Sumudu transform as follows:

$$\frac{1}{c}\int_0^\infty e^{-\frac{t}{u}}\frac{\partial^3 f(x,y,t)}{\partial t^3}dx=\frac{1}{c^3}F(x,y,c)-\frac{1}{c^3}f(x,y,0)-\frac{1}{c^2}\frac{\partial f(x,y,0)}{\partial t}-\frac{1}{c}\frac{\partial^2(x,y,0)}{\partial t^2}$$

4. The triple Laplace transform was defined by the following form:

$$\mathscr{L}_3[f(x,y,t);(p,s,k)]=F(p,s,k)=\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+sy+kt)}f(x,y,t)dxdydt$$

5. The triple Laplace transform for the third-order partial derivative with respect to x can be given by:

$$\mathscr{L}_3[\frac{\partial^3 f(x,y,t)}{\partial x^3};(p,s,k)]=p^3F(p,s,k)-p^2F(0,y,t)-p\frac{\partial F(0,y,t)}{\partial x}-\frac{\partial^2 F(o,y,t)}{\partial x^2}$$

6. By the same way, the triple Laplace transform for the third-order partial derivative with respect to y can be given by

$$\mathscr{L}_3[\frac{\partial^3 f(x,y,t)}{\partial y^3};(p,s,k)]=s^3F(p,s,k)-s^2F(x,0,t)-s\frac{\partial F(x,0,t)}{\partial y}-\frac{\partial^2 F(x,0,t)}{\partial y^2}$$

7. By the same way, the triple Laplace transform for the third-order partial derivative with respect to t can be given by

$$\mathscr{L}_3[\frac{\partial^3 f(x,y,t)}{\partial t^3};(p,s,k)]=k^3F(p,s,k)-k^2F(x,y,0)-k\frac{\partial F(x,y,0)}{\partial t}-\frac{\partial^2 F(x,y,0)}{\partial t^2}$$

Results

In this section, we have applied the triple Sumudu and Laplace transforms for solving general linear third- and fourth-orders PDEs.

General Linear Third-Order PDEs

In this section, the following general third-order PDE has been considered.

$$\begin{aligned}&a_1u_x(x,y,t)+a_2u_y(x,y,t)+a_3u_t(x,y,t)+a_4u_{xx}(x,y,t)+a_5u_{yy}(x,y,t)+a_6u_{tt}(x,y,t)+a_7u_{xt}(x,y,t)+\\&a_8u_{yt}(x,y,t)+a_9u_{xxy}(x,y,t)+a_{10}u_{xxt}(x,y,t)+a_{11}u_{yyt}(x,y,t)+a_{12}u_{yyx}(x,y,t)+a_{13}u_{ttx}(x,y,t)+\\&a_{14}u_{tty}(x,y,t)+a_{15}u_{xxx}(x,y,t)+a_{16}u_{yyy}(x,y,t)+a_{17}u_{ttt}(x,y,t)+a_{18}u(x,y,t)=f(x,y,t)\end{aligned}\quad(2)$$

with IC: $u(0,y,t)=e^{-y-t},\quad y,t>0.$

where $a_1,a_2,......,a_{18}$ are constants

and $a_7=a_8=a_9=a_{10}=a_{11}=a_{12}=a_{13}=a_{14}=0$.

Triple Laplace Transform

In this subsection, Laplace transform for solving Equation (2) has been used with assumptions. a7 = a8 = a9 = a10 = a11 = a12 = a13 = a14 = 0 and f**(**x;y;t**)** = 0 as follows

$$
\begin{aligned}
a_1[pF(p,s,k) &- F(0,y,t)] + a_2[sF(p,s,k) - F(x,0,t)] + a_3[kF(p,s,k) \\
&- F(x,y,0)] + a_4[p^2F(p,s,k) - pF(0,y,t) - \frac{\partial F(0,y,t)}{\partial x}] \\
&+ a_5[s^2F(p,s,k) - sF(x,0,t) - \frac{\partial F(x,0,t)}{\partial y}] + a_6[k^2F(p,s,k) \\
&- kF(x,y,0) - \frac{\partial F(x,y,0)}{\partial t}] + a_{15}[p^3F(p,s,k) - p^2F(0,y,t) \\
&- p\frac{\partial F(0,y,t)}{\partial x} - \frac{\partial^2 F(0,x,t)}{\partial x^2}] + a_{16}[s^3F(p,s,k) - s^2F(x,0,t) \\
k &- s\frac{\partial F(x,0,t)}{\partial y} - \frac{\partial^2 F(x,0,t)}{\partial y^2}] + a_{17}[k^3F(p,s,k) - k^2F(x,y,0) \\
&- \frac{\partial F(x,y,0)}{\partial t} - \frac{\partial^2 F(x,y,0)}{\partial t^2}] + a_{18}u = 0 \qquad (3)
\end{aligned}
$$

Now, by taking triple Laplace transform of expanded ICs as follows

$$\mathscr{L}_3[u(0,y,t)] = F(0,y,t) = \frac{1}{(s+1)(k+1)}, \qquad (4)$$

$$\mathscr{L}_3[\frac{\partial u(0,y,t)}{\partial x}] = \frac{\partial F(0,y,t)}{\partial x} = \frac{1}{(s+1)(k+1)}, \qquad (5)$$

$$\mathscr{L}_3[\frac{\partial^2 u(0,y,t)}{\partial x^2}] = \frac{\partial^2 F(0,y,t)}{\partial x^2} = \frac{1}{(s+1)(k+1)}, \qquad (6)$$

$$\mathscr{L}_3[u(x,0,t)] = F(x,0,t) = \frac{1}{(p-1)(k+1)}, \qquad (7)$$

$$\mathscr{L}_3[\frac{\partial u(x,0,t)}{\partial y}] = \frac{\partial F(x,0,t)}{\partial y} = \frac{-1}{(p-1)(k+1)}, \qquad (8)$$

$$\mathscr{L}_3[\frac{\partial^2 u(x,0,t)}{\partial y^2}] = \frac{\partial^2 F(x,0,t)}{\partial y^2} = \frac{1}{(p-1)(k+1)}, \qquad (9)$$

$$\mathscr{L}_3[u(x,y,0)] = F(x,y,0) = \frac{1}{(p-1)(s+1)}, \qquad (10)$$

$$\mathscr{L}_3[\frac{\partial u(x,y,0)}{\partial t}] = \frac{\partial F(x,y,0)}{\partial t} = \frac{-1}{(p-1)(s+1)}, \qquad (11)$$

and,

$$\mathscr{L}_3[\frac{\partial^2 u(x,y,0)}{\partial t^2}] = \frac{\partial^2 F(x,y,0)}{\partial t^2} = \frac{1}{(p-1)(s+1)}. \qquad (12)$$

Substitute equations (2.3) -(2.11) in Equation (3) to obtain the following

$$
\begin{aligned}
F(p,s,k) \quad (& \quad a_1 p + a_2 s + a_3 k + a_4 p^2 + a_5 s^2 + a_6 k^2 + a_{15} p^3 + a_{16} s^2 \\
+ & \quad a_{17} k^2 + a_{18}) \\
= & \quad \frac{a_1 + a_4 p + a_4 + a_{15} p^2 + a_{15} p + a_{15}}{(s+1)(k+1)} \\
+ & \quad \frac{a_2 + a_5 s - a_5 + a_{16} s^2 - a_{16} s + a_{16}}{(p-1)(k+1)} \\
+ & \quad \frac{a_3 + a_6 k - a_6 + a_{17} k^2 - a_{17} k + a_{17}}{(p-1)(s+1)}
\end{aligned}
$$

In particular, if *a*1 = *a*2 = *a*3 = *a*4 = *a*5 = *a*6 = *a*15 = *a*161 = *a*17 -*a*18 = 1; we obtain

$$F(p,s,k)(f(x,y,t)) = \frac{f(x,y,t)}{(p-1)(s+1)(k+1)},$$

Where

$$f(x,y,t) = p^3 + p^2 + p + s^3 + s^2 + s + k^3 + k^2 + k - 1,$$

Then,

$$F(p,s,k) \quad = \quad \frac{1}{(p-1)(s+1)(k+1)}. \qquad (13)$$

Using inverse of Laplace transform for Equation (13), we obtain the following solution $u(x,y,t) = e^{x-y-t}$.

Example 1. Consider the following third-order PDE

$$u_{xxx}(x,y,t) = 2u_{xx}(x,y,t) + u_x(x,y,t) - u_y(x,y,t) - 3u(x,y,t), \quad x,y,t > 0. \qquad (14)$$

With IC: $u(0,y,t) = e^{-y-t}, \quad y,t > 0.$

BC: $u(x,0,t) = e^{x-t}, \quad x,t > 0.$

Using Laplace transform for Equation (14)

$$
\begin{aligned}
p^3 F(p,s,k) \quad - & \quad p^2 F(0,y,t) - p\frac{\partial F(0,y,t)}{\partial x} - \frac{\partial^2 F(0,x,t)}{\partial x^2} \\
= & \quad 2[p^2 F(p,s,k) - pF(0,y,t) - \frac{\partial F(0,y,t)}{\partial x}] + pF(p,s,k) \\
- & \quad pF(0,y,t) - sF(p,s,k) + F(x,0,t) - 3F(p,s,k).
\end{aligned}
$$

Where

$$\begin{aligned}
\mathscr{L}_3[u(0,y,t)] = F(0,y,t) &= \frac{1}{(s+1)(k+1)},\\
\mathscr{L}_3[\frac{\partial u(0,y,t)}{\partial x}] &= \frac{\partial F(0,y,t)}{\partial x} = \frac{1}{(s+1)(k+1)},\\
\mathscr{L}_3[\frac{\partial^2 u(0,y,t)}{\partial x^2}] &= \frac{\partial^2 F(0,y,t)}{\partial x^2} = \frac{1}{(s+1)(k+1)},\\
\mathscr{L}_3[u(x,0,t)] &= F(x,0,t) = \frac{1}{(p-1)(k+1)},
\end{aligned}$$

Then,

$$\begin{aligned}
F(p,s,k)(p^3-2p-p+s+3) &= \frac{p^3-2p-p+s+3}{(p-1)(s+1)(k+1)},\\
F(p,s,k) &= \frac{1}{(p-1)(s+1)(k+1)}. \qquad (15)
\end{aligned}$$

Taking inverse of Laplace transform for Equation (15) to obtain the following solution:

Triple Sumudu Transform

In this subsection, the triple Sumudu transform for solving PDEs with three variables Equation has been used (2) with assumption . $a7 = a8 = a9 = a10 = a11 = a12 = a13 = a14 = 0$ and $f(x;y;t) = 0$ as follows:

$$\begin{aligned}
a_1 & [\frac{G(u,v,c)-G(0,y,t)}{u}] + a_2[\frac{G(u,v,c)-G(x,0,t)}{v}]\\
+ & a_3[\frac{G(u,v,c)-G(x,y,0)}{c}] + a_4[\frac{G(u,v,c)-G(0,y,t)}{u^2} - \frac{\frac{\partial G(0,y,t)}{\partial x}}{u}]\\
+ & a_5[\frac{G(u,v,c)-G(x,0,t)}{v^2} - \frac{\frac{\partial G(x,0,t)}{\partial y}}{v}] + a_6[\frac{G(u,v,c)-G(x,y,0)}{c^2}\\
- & \frac{\frac{\partial G(x,y,0)}{\partial t}}{c}] + a_{15}[\frac{G(u,v,c)-G(0,y,t)}{u^3} - \frac{\frac{\partial G(0,y,t)}{\partial x}}{u^2} - \frac{\frac{\partial^2 G(0,y,t)}{\partial x^2}}{u}]\\
+ & a_{16}[\frac{G(u,v,c)-G(x,0,t)}{v^3} - \frac{\frac{\partial G(x,0,t)}{\partial y}}{v^2} - \frac{\frac{\partial^2 G(x,0,t)}{\partial y^2}}{v}]\\
+ & a_{17}[\frac{G(u,v,c)-G(x,y,0)}{c^3} - \frac{\frac{\partial G(x,y,0)}{\partial t}}{c^2} - \frac{\frac{\partial^2 G(x,t,0)}{\partial t^2}}{c}]. \qquad (16)
\end{aligned}$$

Taking triple Sumudu transform of ICs

$$S_3[u(0,y,t)] = G(0,y,t) = \frac{1}{(1+v)(1+c)}, \quad (17)$$

$$S_3[\frac{\partial u(0,y,t)}{\partial x}] = \frac{\partial G(0,y,t)}{\partial x} = \frac{1}{(1+v)(1+c)}, \quad (18)$$

$$S_3[\frac{\partial^2 u(0,y,t)}{\partial x^2}] = \frac{\partial^2 G(0,y,t)}{\partial x^2} = \frac{1}{(1+v)(1+c)}, \quad (19)$$

$$S_3[u(x,0,t)] = G(x,0,t) = \frac{1}{(1-u)(1+c)}, \quad (20)$$

$$S_3[\frac{\partial u(x,0,t)}{\partial y}] = \frac{\partial G(x,0,t)}{\partial y} = \frac{-1}{(1-u)(1+c)}, \quad (21)$$

$$S_3[\frac{\partial^2 u(x,0,t)}{\partial y^2}] = \frac{\partial^2 G(x,0,t)}{\partial y^2} = \frac{1}{(1-u)(1+c)}, \quad (22)$$

$$S_3[u(x,y,0)] = G(x,y,0) = \frac{1}{(1-u)(1+v)}, \quad (23)$$

$$S_3[\frac{\partial u(x,y,0)}{\partial t}] = \frac{\partial G(x,y,0)}{\partial t} = \frac{-1}{(1-u)(1+v)}, \quad (24)$$

and,

$$S_3[\frac{\partial^2 u(x,y,0)}{\partial t^2}] = \frac{\partial^2 G(x,y,0)}{\partial t^2} = \frac{1}{(1-u)(1+v)}. \quad (25)$$

Substitute equations (2.22) -(2.30) into Equation (16) to obtain the following

$$\begin{aligned} G(u,v,c) \quad (& a_1u^2v^3c^3 + a_2u^3v^2c^3 + a_3u^3v^3c^2 + a_4uv^3c^3 + a_5u^3vc^3 + a_6u^3v^3c \\ + & a_{15}v^3c^3 + a_{16}u^3c^3 + a_{17}u^3v^3 + a_{17}u^3v^3c^3) \\ = & \frac{a_1u^2v^3c^3 + a_4uv^3c^3 + a_4u^2v^3c^3 + a_{15}v^3c^3 + a_{15}uv^3c^3 + a_{15}u^2v^3c^3}{(1+v)(1+c)} \\ + & \frac{a_2u^3v^2c^3 + a_5u^3vc^3 - a_5u^3v^2c^3 + a_{16}u^3c^3 - a_{16}u^3vc^3 + a_{16}u^3v^2c^3}{(1-u)(1+c)} \\ + & \frac{a_3u^3v^3c^2 + a_6u^3v^3c - a_6u^3v^3c^2 + a_{17}u^3v^3 - a_{17}u^3v^3c + a_{17}u^3v^3c^2}{(1-u)(1+v)}. \end{aligned}$$

In particular, if

$$a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_{15} = a_{16} = a_{17} - a_{18} = 1,$$

Where

$$G(u,v,c)(f(u,v,c) = \frac{f(u,v,c)}{(1-u)(1+v)(1+c)},$$

$$f(u,v,c) = u^2v^3c^3 + u^3v^2c^3 + u^3v^3c^2 + uv^3c^3 + u^3vc^3 + u^3v^3c + v^3c^3 + u^3c^3 + u^3v^3 + u^3v^3c^3$$

$$G(u,v,c) = \frac{1}{(p-1)(s+1)(k+1)}. \quad (26)$$

Using inverse of Sumudu transform for Equation (26) to obtain the following solution:

$$u(x,y,t) = e^{x-y-t}$$

Example 2. Consider the following third-order PDE

$$\begin{aligned} u_{xxx}(x,y,t) &= 2u_{xx}(x,y,t)+u_x(x,y,t)-u_y(x,y,t) \\ &- 3u(x,y,t), \quad x,y,t>0. \end{aligned} \tag{27}$$

With IC $u(0,y,t)=e^{-y-t}, \quad y,t>0.$
BC: $u(x,0,t)=e^{x-t}, \quad x,t>0.$
Taking Sumudu transform for Equation (27), to obtain the following solution:

$$\begin{aligned} \frac{G(u,v,c)-G(0,y,t)}{u^3} &- \frac{\frac{\partial G(0,y,t)}{\partial x}}{u^2}-\frac{\frac{\partial^2 G(0,y,t)}{\partial x^2}}{u}, \\ &= 2[\frac{G(u,v,c)-G(0,y,t)}{u^2}-\frac{\frac{\partial G(0,y,t)}{\partial x}}{u}]+\frac{G(u,v,c)-G(0,y,t)}{u}, \\ &- \frac{G(u,v,c)-G(x,0,t)}{v}-3G(u,v,c). \end{aligned}$$

Where,

$$\begin{aligned} G(0,y,t) &= S_3[u(0,y,t)], \\ \frac{\partial G(0,y,t)}{\partial x} &= S_3[\frac{\partial u(0,y,t)}{\partial x}], \\ \frac{\partial^2 G(0,y,t)}{\partial x^2} &= S_3[\frac{\partial^2 G(0,y,t)}{\partial x^2}], \\ G(x,0,t) &= S_3[u(x,0,t)]. \end{aligned}$$

Hence,

$$\begin{aligned} G(u,v,c)(v-2uv-u^2v+u^3+3u^3v) &= \frac{v-2uv-u^2v+u^3+3u^3v}{(1-u)(1+v)(1+c)}, \\ G(u,v,c) &= \frac{1}{(1-u)(1+v)(1+c)}. \end{aligned} \tag{28}$$

Using Sumudu inverse transform for Equation (28) to obtain the following solution:

$$u(x,y,t)=e^{x-y-t}.$$

General Linear Fourth-Order PDEs
In this section, the following general fourth-order PDE has been considered

$$\begin{aligned} &a_1u_x(x,y,t)+a_2u_y(x,y,t)+a_3u_t(x,y,t)+a_4u_{xx}(x,y,t)+a_5u_{yy}(x,y,t)+a_6u_{tt}(x,y,t)+a_7u_{xt}(x,y,t)+ \\ &a_8u_{yt}(x,y,t)+a_9u_{xxy}(x,y,t)+a_{10}u_{xxt}(x,y,t)+a_{11}u_{yyt}(x,y,t)+a_{12}u_{yyx}(x,y,t)+a_{13}u_{ttx}(x,y,t)+ \\ &a_{14}u_{tty}(x,y,t)+a_{15}u_{xxx}(x,y,t)+a_{16}u_{yyy}(x,y,t)+a_{17}u_{ttt}(x,y,t)+a_{18}u_{xxxt}(x,y,t)+a_{19}u_{xxxy}(x,y,t)+ \\ &a_{20}u_{yyyx}(x,y,t)+a_{21}u_{yyyt}(x,y,t)+a_{22}u_{tttx}(x,y,t)+a_{23}u_{ttty}(x,y,t)+a_{24}u_{xxxx}(x,y,t)+a_{25}u_{yyyy}(x,y,t) \\ &+a_{26}u_{tttt}(x,y,t)+a_{27}u(x,y,t)=f(x,y,t). \end{aligned} \tag{29}$$

Where, $a_1, a_2, \ldots\ldots, a_{27}$ are constants.

with IC: $u(x,y,0) = e^{x-y}, \quad x,y > 0.$

BC: $u(0,y,t) = e^{-y-t}, \quad u(x,0,t) = e^{x-t}, \quad x,y,t > 0.$

Laplace Transform

In this subsection, we use triple Laplace transform for solving Equation (29) with the following assumptions. a7 = a8 = a9 = a10 = a11 = a12 = a13 = a14 = 0 = a18 = a19 = a20 = a21 = a22 = a23 = 0 and f(x;y;t) = 0 as follows

$$\begin{aligned} a_1 & \quad [pF(p,s,k) - F(0,y,t)] + a_2[sF(p,s,k) - F(x,0,t)] + a_3[kF(p,s,k) \\ - & \quad F(x,y,0)] + a_4[p^2F(p,s,k) - pF(0,y,t) - \frac{\partial F(0,y,t)}{\partial x}] + a_5[s^2F(p,s,k) \\ - & \quad sF(x,0,t) - \frac{\partial F(x,0,t)}{\partial y}] + a_6[k^2F(p,s,k) - kF(x,y,0) - \frac{\partial F(x,y,0)}{\partial t}] \\ + & \quad a_{15}[p^3F(p,s,k) - p^2F(0,y,t) - p\frac{\partial F(0,y,t)}{\partial x} - \frac{\partial^2 F(0,x,t)}{\partial x^2}] \end{aligned} \tag{30}$$

$$\begin{aligned} + & \quad a_{16}[s^3F(p,s,k) - s^2F(x,0,t) - s\frac{\partial F(x,0,t)}{\partial y} - \frac{\partial^2 F(x,0,t)}{\partial y^2}] \\ + & \quad a_{17}[k^3F(p,s,k) - k^2F(x,y,0) - k\frac{\partial F(x,y,0)}{\partial t} - \frac{\partial^2 F(x,y,0)}{\partial t^2}] \\ + & \quad a_{24}[p^4F(p,s,k) - p^3F(0,y,t) - p^2\frac{\partial F(0,y,t)}{\partial x} - p\frac{\partial^2 F(0,x,t)}{\partial x^2} \\ - & \quad \frac{\partial^3 F(0,x,t)}{\partial x^3}] + a_{25}[s^4F(p,s,k) - s^3F(x,0,t) - s^2\frac{\partial F(x,0,t)}{\partial y} \\ - & \quad s\frac{\partial^2 F(x,0,t)}{\partial y^2} - \frac{\partial^3 F(x,0,t)}{\partial y^3}] + a_{26}[k^4F(p,s,k) - k^3F(x,y,0) \\ - & \quad k^3\frac{\partial F(x,y,0)}{\partial t} - k\frac{\partial^2 F(x,y,0)}{\partial t^2} - \frac{\partial^3 F(x,y,0)}{\partial t^3}] + a_{27}u = 0. \end{aligned} \tag{31}$$

Taking triple Laplace transform of ICs

$$\mathscr{L}_3[u(0,y,t)] = F(0,y,t) = \frac{1}{(s+1)(k+1)}, \tag{32}$$

$$\mathscr{L}_3[\frac{\partial u(0,y,t)}{\partial x}] = \frac{\partial F(0,y,t)}{\partial x} = \frac{1}{(s+1)(k+1)}, \tag{33}$$

$$\mathscr{L}_3[\frac{\partial^2 u(0,y,t)}{\partial x^2}] = \frac{\partial^2 F(0,y,t)}{\partial x^2} = \frac{1}{(s+1)(k+1)}, \tag{34}$$

$$\mathscr{L}_3[\frac{\partial^3 u(0,y,t)}{\partial x^3}] = \frac{\partial^3 F(0,y,t)}{\partial x^3} = \frac{1}{(s+1)(k+1)}, \tag{35}$$

So,

$$\mathscr{L}_3[u(x,0,t)] = F(x,0,t) = \frac{1}{(p-1)(k+1)}, \quad (36)$$

$$\mathscr{L}_3[\frac{\partial u(x,0,t)}{\partial y}] = \frac{\partial F(x,0,t)}{\partial y} = \frac{-1}{(p-1)(k+1)}, \quad (37)$$

$$\mathscr{L}_3[\frac{\partial^2 u(x,0,t)}{\partial y^2}] = \frac{\partial^2 F(x,0,t)}{\partial y^2} = \frac{1}{(p-1)(k+1)}, \quad (38)$$

$$\mathscr{L}_3[\frac{\partial^3 u(x,0,t)}{\partial y^3}] = \frac{\partial^3 F(x,0,t)}{\partial y^3} = \frac{-1}{(p-1)(k+1)}, \quad (39)$$

Where,

$$\mathscr{L}_3[u(x,y,0)] = F(x,y,0) = \frac{1}{(p-1)(s+1)}, \quad (40)$$

$$\mathscr{L}_3[\frac{\partial u(x,y,0)}{\partial t}] = \frac{\partial F(x,y,0)}{\partial t} = \frac{-1}{(p-1)(s+1)}, \quad (41)$$

$$\mathscr{L}_3[\frac{\partial^2 u(x,y,0)}{\partial t^2}] = \frac{\partial^2 F(x,y,0)}{\partial t^2} = \frac{1}{(p-1)(s+1)}, \quad (42)$$

Substitute equations (3.3) -(3.14) into Equation (30) to obtain the following.
And,

$$\mathscr{L}_3[\frac{\partial^2 u(x,y,0)}{\partial t^3}] = \frac{\partial^3 F(x,y,0)}{\partial t^3} = \frac{-1}{(p-1)(s+1)}. \quad (43)$$

$$\begin{aligned} F(p,s,k) \quad & (\quad a_1p + a_2s + a_3k + a_4p^2 + a_5s^2 + a_6k^2 + a_{15}p^3 + a_{16}s^2 \\ & + \quad a_{17}k^2 + a_{18}) \\ & = \quad \frac{a_1 + a_4p + a_4 + a_{15}p^2 + a_{15}p + a_{15} + a_{24}p^3 + a_{24}p^2 + a_{24}p + a_{24}}{(s+1)(k+1)} \\ & + \quad \frac{a_2 + a_5s - a_5 + a_{16}s^2 - a_{16}s + a_{16} + a_{25}s^3 - a_{25}s^2 + a_{25}s - a_{25}}{(p-1)(k+1)} \\ & + \quad \frac{a_3 + a_6k - a_6 + a_{17}k^2 - a_{17}k + a_{17} + a_{26}k^3 - a_{26}k^2 + a_{26}k - a_{26}}{(p-1)(s+1)} \end{aligned}$$

$$\begin{aligned} F(p,s,k) \quad & (\quad a_1p + a_2s + a_3k + a_4p^2 + a_5s^2 + a_6k^2 + a_{15}p^3 + a_{16}s^2 \\ & + \quad a_{17}k^2 + a_{18}) \\ & = \quad \frac{a_1 + a_4p + a_4 + a_{15}p^2 + a_{15}p + a_{15} + a_{24}p^3 + a_{24}p^2 + a_{24}p + a_{24}}{(s+1)(k+1)} \\ & + \quad \frac{a_2 + a_5s - a_5 + a_{16}s^2 - a_{16}s + a_{16} + a_{25}s^3 - a_{25}s^2 + a_{25}s - a_{25}}{(p-1)(k+1)} \\ & + \quad \frac{a_3 + a_6k - a_6 + a_{17}k^2 - a_{17}k + a_{17} + a_{26}k^3 - a_{26}k^2 + a_{26}k - a_{26}}{(p-1)(s+1)} \end{aligned}$$

$$F(p,s,k)(p,s,k) = \frac{(p,s,k)}{(p-1)(s+1)(k+1)},$$

In particular, if *a*1 = *a*2 = *a*3 = *a*4 = *a*5 = *a*6 = *a*15 = *a*16 = *a*17 = *a*24 = *a*25 = *a*26-14*a*18 = 1

$$f(p,s,k)=p^4+p^3+p^2+p+s^4+s^3+s^2+s+k^4+k^3+k^2+k-4$$

$$F(p,s,k) = \frac{1}{(p-1)(s+1)(k+1)}. \quad (44)$$

Using Laplace inverse transform for Equation (44) to obtain the following solution:

$$u(x,y,t)=e^{x-y-t}.$$

Example 3. Consider the following Fourth-order PDE

$$u_{xxxx}(x,y,t) = -u_{yyy}(x,y,t), \quad x,y,t>0. \quad (45)$$

with IC: $u(0,y,t)=e^{-y-t}, y,t>0.$
BC: $u(x,0,t)=e^{x-t}, x,t>0.$
Using Laplace transform for Equation (45), to obtain the following solution:

$$\begin{aligned} p^4F(p,s,k) &- p^3F(0,y,t)-p^2\frac{\partial F(0,y,t)}{\partial x}-p\frac{\partial^2 F(0,x,t)}{\partial x^2}-\frac{\partial^3 F(0,x,t)}{\partial x^3}] \\ &= s^3F(p,s,k)-s^2F(x,0,t)-p\frac{\partial F(x,0,t)}{\partial v}-\frac{\partial^2 F(x,y,t)}{\partial y^2}. \end{aligned}$$

$$\begin{aligned} \mathscr{L}_3[u(0,y,t)] = F(0,y,t) &= \frac{1}{(s+1)(k+1)}, \\ \mathscr{L}_3[\frac{\partial u(0,y,t)}{\partial x}] = \frac{\partial F(0,y,t)}{\partial x} &= \frac{1}{(s+1)(k+1)}, \end{aligned} \quad (46)$$

$$\begin{aligned} \mathscr{L}_3[\frac{\partial^2 u(0,y,t)}{\partial x^2}] &= \frac{\partial^2 F(0,y,t)}{\partial x^2} = \frac{1}{(s+1)(k+1)}, \\ \mathscr{L}_3[\frac{\partial^3 u(0,y,t)}{\partial x^3}] &= \frac{\partial^3 F(0,y,t)}{\partial x^3} = \frac{1}{(s+1)(k+1)}, \\ \mathscr{L}_3[u(x,0,t)] &= F(x,0,t) = \frac{1}{(p-1)(k+1)}, \\ \mathscr{L}_3[\frac{\partial u(x,0,t)}{\partial y}] &= \frac{\partial F(x,0,t)}{\partial x} = \frac{-1}{(p-1)(k+1)}, \\ \mathscr{L}_3[\frac{\partial^2 u(x,0,t)}{\partial y^2}] &= \frac{\partial^2 F(x,0,t)}{\partial y^2} = \frac{1}{(p-1)(k+1)}. \end{aligned}$$

$$\begin{aligned} F(p,s,k)(p^4+s^3) &= \frac{p^4+s^3}{(p-1)(s+1)(k+1)} \\ F(p,s,k) &= \frac{1}{(p-1)(s+1)(k+1)} \end{aligned} \quad (47)$$

Taking Laplace inverse transform for Equation (47) to obtain the following solution:

$$u(x,y,t)=e^{x-y-t}.$$

Triple Sumudu Transform

In this subsection, we use triple Sumudu transform for solving fourth-order PDEs in general in Equation (29) with the following assumptions . a7 = a8 = a9 = a10 = a11 = a12 = a13 =a14 = 0 = a18 = a19 = a20 = a21 = a22 = a23 = 0 and f (x;y;t) = 0 as follows:

$$
\begin{aligned}
& a_1 \quad [\frac{G(u,v,c)-G(0,y,t)}{u}]+a_2[\frac{G(u,v,c)-G(x,0,t)}{v}] \\
& + \quad a_3[\frac{G(u,v,c)-G(x,y,0)}{c}]+a_4[\frac{G(u,v,c)-G(0,y,t)}{u^2}-\frac{\frac{\partial G(0,y,t)}{\partial x}}{u}] \\
& + \quad a_5[\frac{G(u,v,c)-G(x,0,t)}{v^2}-\frac{\frac{\partial G(x,0,t)}{\partial y}}{v}]+a_6[\frac{G(u,v,c)-G(x,y,0)}{c^2} \\
& - \quad \frac{\frac{\partial G(x,y,0)}{\partial t}}{c}]+a_{15}[\frac{G(u,v,c)-G(0,y,t)}{u^3}-\frac{\frac{\partial G(0,y,t)}{\partial x}}{u^2}-\frac{\frac{\partial^2 G(0,y,t)}{\partial x^2}}{u}] \\
+ \quad & a_{16}[\frac{G(u,v,c)-G(x,0,t)}{v^3}-\frac{\frac{\partial G(x,0,t)}{\partial y}}{v^2}-\frac{\frac{\partial^2 G(x,0,t)}{\partial y^2}}{v}] \\
+ \quad & a_{17}[\frac{G(u,v,c)-G(x,y,0)}{c^3}-\frac{\frac{\partial G(x,y,0)}{\partial t}}{c^2}-\frac{\frac{\partial^2 G(x,y,0)}{\partial t^2}}{c}] \\
+ \quad & a_{24}[\frac{G(u,v,c)-G(0,y,t)}{u^4}-\frac{\frac{\partial G(0,y,t)}{\partial x}}{u^3}-\frac{\frac{\partial^2 G(0,y,t)}{\partial x^2}}{u^2}+\frac{\frac{\partial^3 G(0,y,t)}{\partial x^3}}{u}] \\
+ \quad & a_{25}[\frac{G(u,v,c)-G(x,0,t)}{v^3}-\frac{\frac{\partial G(x,0,t)}{\partial y}}{v^2}-\frac{\frac{\partial^2 G(x,0,t)}{\partial y^2}}{v^2}-\frac{\frac{\partial^3 G(x,0,t)}{\partial y^3}}{v}] \\
+ \quad & a_{26}[\frac{G(u,v,c)-G(x,y,0)}{c^3}-\frac{\frac{\partial G(x,y,0)}{\partial t}}{c^2}-\frac{\frac{\partial^2 G(x,y,0)}{\partial t^2}}{c^2} \\
- \quad & \frac{\frac{\partial^3 G(x,y,0)}{\partial t^3}}{c}]+a_{27}(u,v,c)=0. \qquad (48)
\end{aligned}
$$

Applying triple Sumudu transform of ICs

$$S_3[u(0,y,t)] = G(0,y,t) = \frac{1}{(1+v)(1+c)}, \qquad (49)$$

$$S_3[\frac{\partial u(0,y,t)}{\partial x}] = \frac{\partial G(0,y,t)}{\partial x} = \frac{1}{(1+v)(1+c)}, \qquad (50)$$

$$S_3[\frac{\partial^2 u(0,y,t)}{\partial x^2}] = \frac{\partial^2 G(0,y,t)}{\partial x^2} = \frac{1}{(1+v)(1+c)}, \qquad (51)$$

$$S_3[\frac{\partial^3 u(0,y,t)}{\partial x^3}] = \frac{\partial^3 G(0,y,t)}{\partial x^3} = \frac{1}{(1+v)(1+c)}, \qquad (52)$$

$$S_3[u(x,0,t)] = G(x,0,t) = \frac{1}{(1-u)(1+c)}, \qquad (53)$$

$$S_3[\frac{\partial u(x,0,t)}{\partial y}] = \frac{\partial G(x,0,t)}{\partial y} = \frac{-1}{(1-u)(1+c)}, \qquad (54)$$

$$S_3[\frac{\partial^2 u(x,0,t)}{\partial y^2}] = \frac{\partial^2 G(x,0,t)}{\partial y^2} = \frac{1}{(1-u)(1+c)}, \qquad (55)$$

$$S_3[\frac{\partial^3 u(x,0,t)}{\partial y^3}] = \frac{\partial^3 G(x,0,t)}{\partial y^3} = \frac{-1}{(1-u)(1+c)} \qquad (56)$$

$$S_3[u(x,y,0)] = G(x,y,0) = \frac{1}{(1-u)(1+v)}, \qquad (57)$$

$$S_3[\frac{\partial u(x,y,0)}{\partial t}] = \frac{\partial G(x,y,0)}{\partial t} = \frac{-1}{(1-u)(1+v)}, \qquad (58)$$

$$S_3[\frac{\partial^2 u(x,y,0)}{\partial t^2}] = \frac{\partial^2 G(x,y,0)}{\partial t^2} = \frac{1}{(1-u)(1+v)}, \quad (59)$$

and

$$S_3[\frac{\partial^3 u(x,y,0)}{\partial t^3}] = \frac{\partial^3 G(x,y,0)}{\partial t^3} = \frac{-1}{(1-u)(1+v)}. \quad (60)$$

Substitute equations (3.21) -(2.32) into Equation (48) to obtain the following solution:

$$\begin{aligned} G(u,v,c) & (a_1u^3v^4c^4 + a_2u^4v^3c^4 + a_3u^4v^4c^3 + a_4u^2v^4c^4 + a_5u^4v^2c^4 + a_6u^4v^4c^2 \\ & + a_{15}uv^4c^4 + a_{16}u^4vc^4 + a_{17}u^4v^4c + a_{24}v^4c^4 + a_{25}u^4c^4, \\ & + a_{26}u^4v^4 + a_{27}u^4v^4c^4) \\ & = \frac{a_1u^3v^4c^4 + a_4u^2v^4c^4 + a_4u^3v^4c^4 + a_{15}uv^4c^4 + a_{15}u^2v^4c^4 + a_{15}u^3v^4c^4}{(1+v)(1+c)} \\ & + \frac{a_{24}v^4c^4 + a_{24}uv^4c^4 + a_{24}u^2v^4c^4 + a_{24}u^3v^4c^4}{(1+v)(1+c)} \\ & + \frac{a_2u^4v^3c^4 + a_5u^4v^2c^4 - a_5u^4v^3c^4 + a_{16}u^4vc^4 - a_{16}u^4v^2c^4 + a_{16}u^4v^3c^4}{(1-u)(1+c)} \\ & + \frac{a_{25}u^4c^4 - a_{25}u^4vc^4 + a_{25}u^4v^2c^4 - a_{25}u^4v^3c^4}{(1-u)(1+c)} \\ & + \frac{a_3u^4v^4c^3 + a_6u^4v^4c^2 - a_{26}u^4v^4c^3 + a_{17}u^4v^4c - a_{17}u^4v^4c^2 + a_{17}u^4v^4c^3}{(1-u)(1+v)} \\ & + \frac{a_{26}u^4v^4 - a_{26}u^4v^4c + a_6u^4v^4c^2 - a_{26}u^4v^4c}{(1-u)(1+v)}. \end{aligned}$$

In particular, if $a1 = a2 = a3 = a4 = a5 = a6 = a15 = a161 = a17a24 = a25 = a26$ - $14a18 = 1$, then,

$$\begin{aligned} f(u,v,c) &= u^3v^4c^4 + u^4v^3c^4 + u^4v^4c^3 + u^2v^4c^4 + u^4v^2c^4 + u^4v^4c^2 + uv^4c^4. \\ &+ u^4vc^4 + u^4v^4c + v^4c^4 + u^4c^4 + u^4v^4 - 4u^4v^4c^4 \end{aligned}$$

Where $G(u,v,c)f(u,v,c) = \frac{f(u,v,c)}{(1-u)(1+v)(1+c)}$,

Then, $$G(u,v,c) = \frac{1}{(1-u)(1+v)(1+c)}. \quad (61)$$

Using Sumudu inverse transform for Equation (61) to obtain the following solution $u(x,y,t) = e^{x-y-t}$.

Example 4. Consider the following fourth-order PDE

$$u_{xxxx}(x,y,t) = -u_{yyy}(x,y,t), \qquad x,y,t > 0. \quad (62)$$

With IC $u(0,y,t) = e^{-y-t}, y,t > 0.$

BC: $u(x,0,t) = e^{x-t}, x,t > 0.$

$$\frac{G(u,v,c)-G(0,y,t)}{u^4} - \frac{\frac{\partial G(0,y,t)}{\partial x}}{u^3} - \frac{\frac{\partial^2 G(0,y,t)}{\partial x^2}}{u^2} - \frac{\frac{\partial^3 G(0,y,t)}{\partial x^3}}{u} = \frac{G(u,v,c)-G(x,0,t)}{v^3} - \frac{\frac{\partial G(x,0,t)}{\partial y}}{v^2} - \frac{\frac{\partial^2 G(x,0,t)}{\partial y^2}}{v}.$$

Where,

$$\begin{aligned} G(0,y,t) &= S_3[u(0,y,t)], \\ \frac{\partial G(0,y,t)}{\partial x} &= S_3[\frac{\partial u(0,y,t)}{\partial x}], \\ \frac{\partial^2 G(0,y,t)}{\partial x^2} &= S_3[\frac{\partial^2 u(0,y,t)}{\partial x^2}], \\ \frac{\partial^2 G(0,y,t)}{\partial x^2} &= S_3[\frac{\partial^3 u(0,y,t)}{\partial x^3}], \\ G(0,y,t) &= S_3[u(x,0,t)], \\ \frac{\partial G(x,0,t)}{\partial y} &= S_3[\frac{\partial u(x,0,t)}{\partial y}], \\ \frac{\partial^2 G(x,0,t)}{\partial y^2} &= S_3[\frac{\partial^2 u(x,0,t)}{\partial y^2}]. \end{aligned}$$

$$\begin{aligned} G(u,v,c)(v^3+u^4) &= \frac{v^3+u^4}{(1-u)(1+v)(1+c)}, \\ G(u,v,c) &= \frac{1}{(1-u)(1+v)(1+c)}. \end{aligned} \quad (63)$$

Using Sumudu inverse transform for Equation (63) to obtain the following solution:

$$u(x,y,t) = e^{x-y-t}.$$

Conclusion

In this paper, the triple Sumudu transform has been studied. The properties of the triple Sumudu transform have been derived. The triple Sumudu transform used for solving some PDEs problems. The approximated solutions of these problems using this transform agrees very well with the analytical solutions. As such, this method is more cost effective in terms of computation steps than other existing transform methods. Hence, we can conclude that the new method is computationally very efficient in solving PDEs.

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